

A note on quantum entropy

Frank Hansen

November 30, 2015

Abstract

Incremental information, as measured by the quantum entropy, is increasing when two ensembles are united. This result was proved by Lieb and Ruskai, and it is the foundation for the proof of strong sub-additivity of quantum entropy. We present a truly elementary proof of this fact in the context of the broader family of matrix entropies introduced by Chen and Tropp.

1 Introduction

Let ρ be a positive definite matrix on a bipartite system $H = H_1 \otimes H_2$ of Hilbert spaces H_1 and H_2 of finite dimensions. Lieb and Ruskai [6, Theorem 1] proved that the function

$$\rho \rightarrow S(\rho) - S(\rho_1)$$

is concave in positive definite ρ on H , where $\rho_1 = \text{Tr}_2 \rho$ is the partial trace of ρ on H_1 and $S(\rho) = -\text{Tr} \rho \log \rho$ is the quantum entropy of ρ . The proof used Klein's inequality and Lieb's concavity theorem. Before giving a truly elementary proof of this result we broaden the investigation and consider functions of the form

$$(1) \quad G(\rho) = d_2^{-1} \text{Tr}_{12} f(d_2 \rho) - \text{Tr}_1 f(\rho_1),$$

where $f: (0, \infty) \rightarrow \mathbf{R}$ is a given function, and d_2 is the dimension of H_2 . If f is sufficiently smooth then G is Fréchet differentiable and the first Fréchet differential is given by

$$\begin{aligned} dG(\rho)h &= \text{Tr}_{12} df(d_2 \rho)h - \text{Tr}_1 df(\rho_1)h_1 \\ &= \text{Tr}_{12} f'(d_2 \rho)h - \text{Tr}_1 f'(\rho_1)h_1, \end{aligned}$$

where we used $\text{Tr } df(x)h = \text{Tr } f'(x)h$, see for example [3, Theorem 2.2]. We continue to calculate the second Fréchet differential

$$(2) \quad d^2G(\rho)(h, h) = d_2 \text{Tr } {}_{12} h df'(d_2 \rho) h - \text{Tr } {}_1 h_1 df'(\rho_1) h_1$$

in positive definite ρ and self-adjoint h .

1.1 Matrix entropies

Matrix entropies were introduced by Chen and Tropp [2] as a tool to obtain concentration inequalities for random matrices, and their representing functions may be characterised in various ways [4, Theorem 1.2]. In particular, a twice differentiable function $f: (0, \infty) \rightarrow \mathbf{R}$ is (the representing function) of a matrix entropy if and only if the function of two variables

$$(3) \quad (x, h) \rightarrow \text{Tr } h^* df'(x) h$$

is convex for positive definite ρ and arbitrary h . Lieb [5, Theorem 3] proved that the function

$$(4) \quad (x, h) \rightarrow \text{Tr } h^* d \log(x) h = \int_0^\infty \text{Tr } h^* \frac{1}{\rho + \lambda} h \frac{1}{\rho + \lambda} d\lambda$$

is convex. The function $t \rightarrow t \log t$ is therefore a matrix entropy (this result was obtained by different means in [2]).

The convexity statement in (4) may be easily obtained by the following argument. Consider the positive function

$$k(t, s) = \frac{\log t - \log s}{t - s} = \int_0^1 (\lambda t + (1 - \lambda)s)^{-1} d\lambda \quad t, s > 0,$$

and let L_x and R_x denote left and right multiplication with x . Then

$$\text{Tr } h^* d \log(x) h = \sum_{i,j=1}^n |(he_i | e_j)|^2 \frac{\log \lambda_i - \log \lambda_j}{\lambda_i - \lambda_j} = \text{Tr } h^* k(L_x, R_x) h,$$

where the intermediary calculation is carried out in an orthonormal basis (e_1, \dots, e_n) of eigenvectors of x with corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ counted with multiplicity. Combining the two formulas we obtain

$$\text{Tr } h^* d \log(x) h = \int_0^1 \text{Tr } h^* (\lambda L_x + (1 - \lambda) R_x)^{-1} h d\lambda.$$

The convexity in (4) now follows from convexity of the operator map $(A, B) \rightarrow B^* A^{-1} B$, where A is positive definite, proved by Lieb and Ruskai [7]. With Ando's elegant proof [1, Theorem 1] this result is readily accessible.

The above line of arguments may be generalised, and we obtain that a sufficient condition for a function $f: (0, \infty) \rightarrow \mathbf{R}$ to be a matrix entropy is that the second derivative f'' is positive, decreasing and operator convex [4, Theorem 1.3].

Lemma 1.1. *Let Φ be a conditional expectation, and let f be a matrix entropy. Then*

$$\mathrm{Tr} \Phi(h)^* df'(\Phi(x)) \Phi(h) \leq \mathrm{Tr} h^* df'(x) h$$

for positive definite ρ and arbitrary h .

Proof. In the finite dimensional case that we consider Φ may be written on the form

$$\Phi(x) = \sum_{i=1}^n p_i u_i^* x u_i$$

for unitaries u_1, \dots, u_n and non-negative weights p_1, \dots, p_n summing up to one. The convexity of the map in (3) thus yields

$$\begin{aligned} \mathrm{Tr} \Phi(h)^* df'(\Phi(x)) \Phi(h) &\leq \sum_{i=1}^n p_i \mathrm{Tr} u_i^* h u_i df'(u_i^* x u_i) u_i^* h u_i \\ &= \mathrm{Tr} h^* df'(x) h, \end{aligned}$$

which is the assertion. **QED**

2 The main result

Theorem 2.1. *Let $f: (0, \infty) \rightarrow \mathbf{R}$ be a matrix entropy. The function*

$$G(\rho) = d_2^{-1} \mathrm{Tr}_{12} f(d_2 \rho) - \mathrm{Tr}_1 f(\rho_1)$$

is convex in positive definite ρ acting on a bipartite system $H = H_1 \otimes H_2$ of Hilbert spaces of finite dimensions, where ρ_1 denotes the partial trace of ρ on H_1 and $d_2 = \dim H_2$.

Proof. We may write $\rho_1 \otimes 1_2 = d_2 \pi_1(\rho)$ in terms of a conditional expectation π_1 on $B(H)$. Since f is a matrix entropy we obtain

$$\begin{aligned} \text{Tr}_1 h_1^* df'(\rho_1) h_1 &= d_2^{-1} \text{Tr}_{12} (h_1 \otimes 1_2)^* df'(\rho_1 \otimes 1_2) (h_1 \otimes 1_2) \\ &= d_2 \text{Tr}_{12} \pi_1(h)^* df'(d_2 \pi_1(\rho)) \pi_1(h) \\ &\leq d_2 \text{Tr}_{12} h^* df'(d_2 \rho) h, \end{aligned}$$

where we used Lemma 1.1. It then follows from (2) that the second Fréchet differential $d^2 G(\rho)(h, h)$ is non-negative, so G is convex. **QED**

Example 2.2. If we consider the matrix entropy $f(t) = t \log t$ then the map

$$\begin{aligned} G(\rho) &= d_2^{-1} \text{Tr}_{12} d_2 \rho \log(d_2 \rho) - \text{Tr}_1 \rho_1 \log \rho_1 \\ &= \log d_2 \text{Tr}_{12} \rho + \text{Tr}_{12} \rho \log \rho - \text{Tr}_1 \rho_1 \log \rho_1 \end{aligned}$$

is convex. But this shows that the map $\rho \rightarrow S(\rho) - S(\rho_1)$ is concave, where $S(\rho) = -\text{Tr} \rho \log \rho$ is the von Neumann entropy.

Example 2.3. If we consider the matrix entropy $f(t) = t^p$ for $1 \leq p \leq 2$, it follows that the map

$$G(\rho) = d_2^{p-1} \text{Tr}_{12} \rho^p - \text{Tr}_1 \rho_1^p$$

is convex.

References

- [1] T. Ando. Concavity of certain maps of positive definite matrices and applications to Hadamard products. *Linear Algebra Appl.*, 26:203–241, 1979.
- [2] R.A. Chen and J.A. Tropp. Subadditivity of matrix φ -entropy and concentration of random matrices. *Electron. J. Probab.*, 19(27):1–30, 2014.
- [3] F. Hansen and G.K. Pedersen. Perturbation formulas for traces on C^* -algebras. *Publ. RIMS, Kyoto Univ.*, 31:169–178, 1995.
- [4] F. Hansen and Z. Zhang. Characterisation of matrix entropies. *Lett Math Phys*, 105:1399–1411, 2015.
- [5] E. Lieb. Convex trace functions and the Wigner-Yanase-Dyson conjecture. *Advances in Math.*, 11:267–288, 1973.

- [6] E. Lieb and M.B. Ruskai. Proof of the strong subadditivity of quantum-mechanical entropy. *J. Math. Phys.*, 14:1938–1941, 1973.
- [7] E.H. Lieb and M.B. Ruskai. Some operator inequalities of the Schwarz type. *Adv. in Math.*, 12:269–273, 1974.

Frank Hansen: Institute for Excellence in Higher Education, Tohoku University,
Japan.
Email: frank.hansen@m.tohoku.ac.jp.